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Generalization of the Forchheimer-extended Darcy flow model to the tensor permeability case via a variational principle

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A convex variational principle is used to obtain a generalization of the empirical nonlinear one-dimensional Forchheimer-extended Darcy flow equation to the multidimensional and anisotropic (tensor permeability) case. A modified permeability that is a function of flow velocity (or pressure gradient) is introduced in order to transform the nonlinear flow equation into a pseudo-linear form. Imposing an incompressibility condition on this pseudo-linear equation leads to a flow equation in Euler–Lagrange form which is used to build the corresponding variational principle. It is demonstrated that the variational principle is based on minimizing the power (time rate of doing work) required by the fluid to flow at a certain velocity under a prescribed pressure gradient. A consistent generalization of the Forchheimer equation to the tensor case then follows from the variational principle. The existence and uniqueness of solutions to the nonlinear flow equations might also be demonstrated using the variational principle on a case by case basis, once appropriate boundary conditions are chosen.

1. Introduction

Theoretical progress towards modelling flow in porous media has evolved in an interesting manner. It took around 100 years, after Darcy's findings were published in 1856, for the theory of flow through porous media to parallel and somewhat surpass the knowledge acquired through experimental evidence. For instance, the concept developed by Brinkman (1947), the volume averaging technique introduced by Whitaker (1969) and explored by Bear & Bachmat (1990), the generalized equations proposed by Hsu & Cheng (1990), etc., are examples of fundamental theoretical advances leading to more complex flow models. It is noteworthy that only recently, with the use of the expensive NMR technique, has the 'visualization' of a flow field inside a porous solid matrix became a reality (see, for instance, Givler & Altobelli 1994). Experimentation is catching up with theoretical development, and building fundamental ground for further theoretical advances.

The simplest model for flow through a porous medium is the one-dimensional model derived by Darcy (1856). Obtained from empirical evidence, the Darcy law indicates that for an incompressible fluid flowing through a channel filled with a fixed, uniform, and isotropic porous matrix, the flow speed varies linearly with longitudinal pressure variation. Subsequently, Dupuit (1863) and Forchheimer (1901) presented further empirical evidence that the Darcy law, or the linearity between speed and pressure variation, breaks down for large enough flow speed (a compilation of several experimental results is presented by MacDonald *et al.* 1979).

A heuristic extension of the Forchheimer-extended Darcy model to multidimensional flow was presented by Stanek & Szekely (1974). It is interesting to note that in a recent monograph, Kaviany (1991) pointed out the lack of multidimensional experimental results to support the extension by Stanek & Szekely (1974). He went on to propose a scalar form for the Forchheimer term, similar to the one proposed by Ward (1964), known to be valid for unidirectional flow. That the cubic drag term proposed by Forchheimer (1901) should in fact be quadratic was indicated in the work of MacDonald *et al.* (1979). This was emphasized later by Joseph Nield & Papanicolaou (1982) who stressed also that the (form) drag force modelled by the Forchheimer term acts in a direction opposite to the velocity vector. It follows that in multidimensional flow, the momentum equations for each velocity component derived using the Forchheimer-extended Darcy model are coupled to each other.

Another important topic related to multidimensional flow in porous media is that of isotropy. Invoking isotropy of the medium, in which permeability is invariant with location, is a norm among published work in the area. Any attempt to analyse flow through an anisotropic porous medium using the original Forchheimer-extended Darcy equation is at least speculative.

This note considers the theoretical generalization to the tensor permeability case (anisotropic medium) of the empirically obtained Forchheimer-extended Darcy unidirectional flow model. Initially, a convex variational principle that minimizes the power (time rate of doing work) of the flow is built from the isotropic multidimensional model. This principle is then extended to the tensor permeability case and the result transformed back into a flow equation format.

2. Forchheimer-extended Darcy equation: pseudo-linear form

The empirically obtained unidirectional momentum equation for flow of incompressible fluid through a fully saturated homogeneous and isotropic porous medium (Ward 1964) is

$$-\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mu}{K}v + \frac{\rho c_F}{K^{1/2}}v^2, \tag{2.1}$$

where $\mathrm{d}p/\mathrm{d}x$ is the pressure variation along the flow direction, v is the seepage flow speed, μ is the dynamic viscosity of the fluid, ρ is the density of the fluid, K is the porous matrix permeability, and c_F is the Forchheimer or inertia coefficient.

Equation (2.1) can be extended to vectorial form (see Hsu & Cheng 1990 for a formal derivation of a general equation obtained by volumetric averaging with closure based on drag force due to solid particles), assuming an isotropic and still homogeneous porous medium, as

$$-\frac{1}{\rho}\nabla p = \frac{\nu}{K}\boldsymbol{v} + \frac{c_F}{K^{1/2}}|\boldsymbol{v}|\,\boldsymbol{v}. \tag{2.2}$$

Notice that a basic physical assumption behind (2.2) is the colinearity of form and viscous drag forces. The nonlinear nature of (2.2) can be relegated to the permeability term by introducing a velocity-dependent permeability parameter, or modified permeability $\hat{K}$:

$$\hat{K} = \frac{K}{1+(K^{1/2}c_F/\nu)\,|\boldsymbol{v}|}, \tag{2.3}$$

where ν is the kinematic viscosity of the fluid. So (2.2) is now written in a pseudo-linear form

$$-\frac{1}{\rho}\nabla p = \frac{\nu}{\hat{K}}\boldsymbol{v}. \tag{2.4}$$

Equation (2.4) is similar to the Darcy equation for multidimensional flow, except that the permeability is now velocity dependent. The modified permeability dependence on velocity can be substituted with a pressure gradient dependence by eliminating the velocity from (2.3) and (2.4). Keeping the positive root,

$$\hat{K} = \frac{-1+(1+4\gamma K^{3/2}|\nabla p|)^{1/2}}{2\gamma K^{1/2}|\nabla p|}, \tag{2.5}$$

with $\gamma = c_F/(\nu^2\rho)$. Equation (2.5) indicates that the modified permeability $\hat{K}$ varies with $|\nabla p|^{-1/2}$ for large pressure gradients, as expected. Also, it is easy to verify that the nonlinear velocity effect is negligible when

$$|\nabla p| \ll \frac{1}{\gamma K^{3/2}}. \tag{2.6}$$

It is worth emphasizing that pressure and velocity are fundamental quantities that can both be measured in the field, with permeability being the derived quantity. Therefore (2.4) and (2.5) are also of practical importance because they provide a simpler model to compute the modified permeability that incorporates the nonlinear character of the flow.

3. Flow equation

By combining the vectorial form of the Forchheimer-extended Darcy equation written as in (2.4), with the continuity equation for incompressible fluid

$$\nabla\cdot\boldsymbol{v} = 0, \tag{3.1}$$

the flow equation reduces to

$$\left(\frac{-1}{\rho}\right)\nabla\cdot\frac{\hat{K}}{\nu}\nabla p = 0. \tag{3.2}$$

Substituting the $\hat{K}$ expression given by (2.3) into (3.2) leads to a flow equation in terms of pressure gradient and magnitude of velocity vector:

$$\nabla\cdot\frac{K/\nu}{1+(K^{1/2}c_F/\nu)|\boldsymbol{v}|}\nabla p = 0. \tag{3.3}$$

To solve (3.3), the magnitude of the velocity vector has to be obtained from the Forchheimer-extended Darcy model. Taking the magnitude of both sides of (2.2):

$$\frac{1}{\rho}|\nabla p| = \frac{\nu}{K}|\boldsymbol{v}| + \frac{c_F}{K^{1/2}}|\boldsymbol{v}|^2, \tag{3.4}$$

and solving for $|\boldsymbol{v}|$, keeping in mind that only the positive root is of interest, results in

$$|\boldsymbol{v}| = \frac{\nu}{2c_F K^{1/2}}\{-1+(1+4\gamma K^{3/2}|\nabla p|)^{1/2}\}. \tag{3.5}$$

With (3.5) inserted into (3.3), a nonlinear flow equation in the pressure gradient is obtained:

$$\nabla \cdot \frac{2K/\nu}{1+(1+4\gamma K^{3/2}|\nabla p|)^{1/2}} \nabla p = 0. \tag{3.6}$$

In the following section it is demonstrated that (3.6) is convenient for building a variational principle of the nonlinear Forchheimer-extended Darcy model.

4. A variational principle

A variational principle is now devised as a tool to extend the nonlinear flow equation (3.6), valid for an isotropic and homogeneous porous matrix, to the tensor permeability case. Suppose $G \mid R \to R$ is a continuous function, $\Omega \subset R^n$, $n = 2$ or 3, is smooth and bounded, and $p \mid \Omega \to R$ is continuous. Extrema of the functional

$$I[p] = \int_\Omega G(|\nabla p|)\,\mathrm{d}\Omega \tag{4.1}$$

are to be computed. Letting $s = |\nabla p|$, it is straightforward to show that the Euler–Lagrange equation for the principle is

$$\nabla \cdot \frac{G'}{s} \nabla p = 0, \tag{4.2}$$

where $G' = \mathrm{d}G/\mathrm{d}s$. The nonlinear flow equation (3.6) has a similar form if

$$G'(s) = \frac{2Ks/\nu}{1+(1+4\gamma K^{3/2} s)^{1/2}}. \tag{4.3}$$

Integration of (4.3) in s leads to

$$G(s) = \frac{1}{12\nu\gamma^2 K^2}[(1+4\gamma K^{3/2} s)^{3/2} - 6\gamma K^{3/2} s - 1], \tag{4.4}$$

with the arbitrary constant being determined by matching the result with limiting cases. The variational principle for the Forchheimer-extended Darcy flow equation is thus to minimize

$$I[p] = \frac{1}{12\nu\gamma^2 K^2}\int_\Omega [(1+4\gamma K^{3/2}|\nabla p|)^{3/2} - 6\gamma K^{3/2}|\nabla p| - 1]\,\mathrm{d}\Omega. \tag{4.5}$$

If the pressure gradient is small, i.e. $4\gamma K^{3/2}|\nabla p| \ll 1$, then the integrand approaches $(K/\nu)|\nabla p|^2$, and the variational principle for the linear Darcy flow model is recovered.

In remains to establish the convexity of $I[p]$. Since the second derivative of $G(s)$,

$$G''(s) = \frac{K/\nu}{(1+4\gamma K^{3/2} s)^{1/2}}, \tag{4.6}$$

is always positive, I is convex over $W_0^{1,\infty}(\Omega, R)$ (see Dacorogna 1989). As suggested by one of the reviewers, the existence and uniqueness of solutions to the nonlinear flow equation could be verified once boundary conditions are precisely set. Such effort is not within the scope of the present study because it is particular to each specific boundary value problem.

It remains to be determined what physical quantity is represented by the functional $I[p]$. Notice that (2.4), the pseudo-linear Darcy equation with modified permeability, implies that the vector velocity and pressure gradient are colinear. Equations (3.1) and (4.2) indicate that

$$G'(|\nabla p|)\,\nabla p/|\nabla p| = \boldsymbol{v}. \tag{4.7}$$

Defining the unit vector aligned with ∇p as

$$\hat{\boldsymbol{n}} = \nabla p/|\nabla p|, \tag{4.8}$$

and combining it with (4.7), leads to a function G' as follows:

$$G'(|\nabla p|) = \mathrm{d}G/\mathrm{d}|\nabla p| = |\boldsymbol{v}|. \tag{4.9}$$

Therefore, function $G(|\nabla p|)$ represents, in two dimensions, the power (time rate of doing work) per unit of area (or per unit of volume if in three dimensions) necessary for the fluid to flow at speed $|\boldsymbol{v}|$ under the pressure gradient $|\nabla p|$. The physical interpretation of functional $I[p]$, equation (4.5), is that of the power (quantity to be minimized) associated with $|\boldsymbol{v}|$ and $|\nabla p|$.

5. Extension to tensor permeability

Suppose, as in an anisotropic medium, the permeability be expressed by a symmetric, positive definite 2×2 or 3×3 tensor, $\boldsymbol{K}$, for flow in a set $\Omega \subset R^n$, with $n = 2$ or 3, respectively. What is the appropriate generalization of the Forchheimer-extended Darcy model, equation (2.2)? Direct generalization is not fruitful since there are several possible generalizations (see Bachmat 1965 for such a generalization).

In what follows, it is proposed that the variational principle derived in the previous section for the isotropic case be used as the basis for generalizing the flow equation. Start by rewriting the isotropic scalar variational principle, equation (4.4), as

$$G = \frac{1}{12\nu\gamma^2K^2}\{[1+4\gamma K(\nabla p\cdot K\nabla p)^{1/2}]^{3/2} - 6\gamma K(\nabla p\cdot K\nabla p)^{1/2} - 1\}. \tag{5.1}$$

Equation (5.1) suggests the following generalization for the tensor-permeability case:

$$G = \frac{1}{12\nu\gamma^2\kappa^2}\{[1+4\gamma\kappa(\nabla p\cdot \boldsymbol{K}\nabla p)^{1/2}]^{3/2} - 6\gamma\kappa(\nabla p\cdot \boldsymbol{K}\nabla p)^{1/2} - 1\}, \tag{5.2}$$

with $\kappa^2 = \det\boldsymbol{K}$ for a 2×2 tensor, or $\kappa^2 = (\det\boldsymbol{K})^{2/3}$ for a 3×3 tensor. To analyse this functional, let $s = (\nabla p\cdot\boldsymbol{K}\nabla p)^{1/2}$, so that the variational principle is of the form

$$I[p] = \int_\Omega G(s)\,\mathrm{d}\Omega, \tag{5.3}$$

where

$$G(s) = \frac{1}{12\nu\gamma^2\kappa^2}[(1+4\gamma\kappa s)^{3/2} - 6\gamma\kappa s - 1]. \tag{5.4}$$

The Euler–Lagrange equation is then

$$\nabla\cdot\frac{G'}{s}\boldsymbol{K}\nabla p = 0. \tag{5.5}$$

Computing G' from (5.4), and defining a modified permeability tensor, $\hat{\boldsymbol{K}}$, equation (5.5) can be written in the same form as the flow equation for an isotropic medium, equation (3.2), as

$$\nabla\cdot\frac{\hat{\boldsymbol{K}}}{\nu}\nabla p = 0, \tag{5.6}$$

where
$$\hat{\boldsymbol{K}} = \frac{2\boldsymbol{K}}{1+[1+4\gamma\kappa(\nabla p\cdot\boldsymbol{K}\nabla p)^{1/2}]^{1/2}}. \tag{5.7}$$

Since
$$\boldsymbol{v} = \left(\frac{-1}{\rho}\right)\frac{\hat{\boldsymbol{K}}}{\nu}\nabla p, \tag{5.8}$$

then
$$\hat{\boldsymbol{K}} = \frac{\boldsymbol{K}}{1+(c_F/\nu)\,\kappa(\boldsymbol{v}\cdot\boldsymbol{K}^{-1}\boldsymbol{v})^{1/2}}, \tag{5.9}$$

and finally,
$$\left(\frac{-1}{\rho}\right)\nabla p = \nu\left[1+\frac{c_F}{\nu}\kappa(\boldsymbol{v}\cdot\boldsymbol{K}^{-1}\boldsymbol{v})^{1/2}\right]\boldsymbol{K}^{-1}\boldsymbol{v}. \tag{5.10}$$

Equation (5.10) is the generalization of the Forchheimer-extended Darcy equation to tensor permeability. Note the analogy between (5.10) and the isotropic equation (2.2).

To show convexity of the tensor-permeability functional, it is sufficient to observe that
$$G'' = \frac{1/\nu}{(1+4\gamma\kappa s)^{1/2}} \tag{5.11}$$
is positive.

A special case where (5.10) could be tested against experimental evidence is now proposed. Consider an array of spherical particles whose centres are on a rectangular lattice, but with different interparticle spacings in each direction. With the principal axes of the permeability tensor aligned with a Cartesian system of coordinates, the inverse of the permeability tensor is diagonal with elements: $1/K_x$, $1/K_y$, $1/K_z$. Equation (5.10) prescribes the following function:
$$\left(\frac{-1}{\rho}\right)\nabla p = \nu\left(\frac{v_x}{K_x},\frac{v_y}{K_y},\frac{v_z}{K_z}\right)+c_F(K_xK_yK_z)^{1/3}\left(\frac{v_x^2}{K_x}+\frac{v_y^2}{K_y}+\frac{v_z^2}{K_z}\right)^{1/2}\left(\frac{v_x}{K_x},\frac{v_y}{K_y},\frac{v_z}{K_z}\right). \tag{5.12}$$

Consider further a simplified flow configuration unidirectional in x, with speed v_x. From (5.12) the pressure drop is related to flow speed by
$$\left(\frac{-1}{\rho}\right)\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\nu}{K_x}v_x+\frac{c_F}{K_x^{1/2}}\left(\frac{K_y}{K_x}\frac{K_z}{K_x}\right)^{1/3}v_x^2. \tag{5.13}$$

An interesting limiting case would be that of flow through parallel tubes when K_y/K_x and K_z/K_x are zero. In this case, the nonlinear (form) drag goes to zero, as expected since momentum in this case is diffused by viscous drag only.

6. Extension to a tensor inertia coefficient

We finally point out that the inertia coefficient c_F is representative of the microscopic form drag imposed by the solid porous matrix, that is, it depends on the geometry (shape) of the solid within each representative elementary volume of the porous medium. It is worth noting that c_F and K are independent parameters. It is possible to think of cases in which a porous medium presents different amounts of anisotropy for permeability and inertia (Forchheimer) coefficients. One can consider, for instance, spheres with a small patch of sand roughness attached to their surfaces. If the spheres are uniformly distributed in a rectangular lattice, and as they have essentially the same geometry and diameter, the medium can provide an essentially isotropic viscous drag (permeability) measured at low flow speed (Darcy regime). However, the form drag (inertia coefficient), measured at high flow speed, is expected to be anisotropic as the patch will induce flow separation at different surface locations for different flow directions.

In cases when the inertia coefficient is anisotropic, the γ coefficient in (5.1) must be translated into a tensor, leading to an equation similar to (5.2) written as

$$G = \frac{1}{12\nu\Gamma^2\kappa^2}\{[1+4\Gamma\kappa(\nabla p\cdot\boldsymbol{K}\nabla p)^{1/2}]^{3/2} - 6\Gamma\kappa(\nabla p\cdot\boldsymbol{K}\nabla p)^{1/2} - 1\}, \tag{6.1}$$

with $\Gamma^2 = \det\boldsymbol{\gamma}$ for a 2×2 tensor, or $\Gamma^2 = (\det\boldsymbol{\gamma})^{2/3}$ for a 3×3 tensor. Notice that $\boldsymbol{\gamma}$ is the modified inertia coefficient tensor, that is, $\boldsymbol{\gamma} = [1/(\nu^2\rho)]\,\boldsymbol{c}_F$, where $\boldsymbol{c}_F$ is the inertia coefficient tensor. The analysis following (5.2) still holds for the tensor inertia coefficient case by simply substituting γ by Γ in (5.4), (5.7) and (5.11). Equations (5.9) and (5.10) are rewritten, respectively, as

$$\hat{\boldsymbol{K}} = \frac{\boldsymbol{K}}{1+\nu\rho\Gamma\kappa(\boldsymbol{v}\cdot\boldsymbol{K}^{-1}\boldsymbol{v})^{1/2}}, \tag{6.2}$$

and

$$\left(\frac{-1}{\rho}\right)\nabla p = \nu[1+\nu\rho\Gamma\kappa(\boldsymbol{v}\cdot\boldsymbol{K}^{-1}\boldsymbol{v})^{1/2}]\boldsymbol{K}^{-1}\boldsymbol{v}. \tag{6.3}$$

Equation (6.3) is the generalization of the Forchheimer-extended Darcy equation to the tensor permeability and tensor inertia coefficient.

7. Conclusion

A variational formulation is developed and applied to a flow model based upon the Forchheimer-extended Darcy equation and incompressible fluid constraint. The functional of the variational principle is shown to represent the power (time rate of doing work) necessary for the fluid to flow at a certain speed under a particular pressure gradient. The variational form is then utilized to extend the flow model to an anisotropic case, where the permeability and inertia coefficient parameters have to be substituted by a 2×2 or by a 3×3 tensors. The result is traced back to a consistent Forchheimer-extended Darcy equation for modelling fluid flow of incompressible fluid through anisotropic media. The variational principle can also be used for demonstrating the existence and uniqueness of solutions to the nonlinear flow equations, once boundary conditions appropriate to an specific problem are specified.

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REFERENCES

Bachmat, Y. 1965 Basic transport coefficients as aquifer characteristics. *IASH Symp. on the Hydrology of Fractured Rocks, Dubrovnik*, vol. 1, pp. 63–75.

Bear, J. & Bachmat, Y. 1990 *Introduction to Modeling of Transport Phenomena in Porous Media.* Kluwer.

Brinkman, H. C. 1947 A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Appl. Sci. Res.* A 1, 27–34.

DACOROGNA, B. 1989 *Direct Methods in the Calculus of Variations*. Springer.

DARCY, H. 1856 *Les Fontaines Publiques de la Ville de Dijon*. Victor Dalmond.

DUPUIT, J. 1863 *Estudes Theoriques et Pratiques sur le Mouvement des Eaux*. Dunod.

FORCHHEIMER, P. 1901 Wasserbewegung durch Boden. *VDIZ*. **45**, 1782–1788.

GIVLER, R. C. & ALTOBELLI, S. A. 1994 A determination of the effective viscosity for the Brinkman–Forchheimer flow model. *J. Fluid Mech.* **258**, 355–370.

HSU, C. T. & CHENG, P. 1990 Thermal dispersion in a porous medium. *Intl J. Heat Mass Transfer* **33**, 1587–1597.

JOSEPH, D. D., NIELD, D. A. & PAPANICOLAOU, G. 1982 Nonlinear equation governing flow in a saturated porous medium. *Water Resources Res.* **18**, 1049–1052.

KAVIANY, M. 1991 *Principles of Heat Transfer in Porous Media*. Springer.

MACDONALD, I. F., EL-SAYED, M. S., MOW, K. & DULLIEN, F. A. L. 1979 Flow through porous media: the Ergun equation revisited. *Indust. Chem. Fundam.* **18**, 199–208.

STANEK, V. & SZEKELY, J. 1974 Three-dimensional flow of fluids through nonuniform packed beds. *AIChE J.* **20**, 974–980.

WARD, J. C. 1964 Turbulent flow in porous media. *J. Hydraul. Div. ASCE* **90** (HY5), 1–12.

WHITAKER, S. 1969 Advances in theory of fluid motion in porous media. *Indust. Engng Chem.* **61**, 14–28.